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1978 J. Phys. A: Math. Gen. 11 297

(http://iopscience.iop.org/0305-4470/11/2/009)

J. Phys. A: Math. Gen., Vol. 11, No. 2, 1978. Printed in Great Britain. © 1978

Radiative effects in a plane wave moving along a magnetic field

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Received 21 June 1977, in final form 26 September 1977

Abstract. Radiative effects are considered when a scalar particle interacts with the field of an intense plane wave of a general type propagating along a magnetic field. Within the framework of operator technique the mass operator of a particle is calculated. A detailed analysis in the case of a circularly polarised monochromatic wave is carried out. The behaviour of particles in the vicinity of the cyclotron resonance is analysed.

1. Introduction

In recent years the behaviour of particles in electromagnetic fields of complex configuration has been widely discussed. One field of such a type is given by an electromagnetic plane wave propagating along a magnetic field. The Klein–Gordon and Dirac equations in such a field were solved by Redmond (1965) who also made a simple analysis of the classical motion of a particle. The Green functions of the scalar and spinor particles for this case were found by Batalin and Fradkin (1970) using the functional integration technique. The production of a pair of particles by a photon in such a field was discussed by Oleinik (1971).

In the field configuration under consideration a very interesting resonant situation is realised: at the cyclotron resonance point, where the wave frequency coincides with the cyclotron frequency of motion of a particle in the magnetic field with the Doppler shift taken into account, the energy transfer of a particle to a wave and *vice versa* can take place, and this may be used in physical applications. One such application was discussed by Kolomensky and Lebedev (1963), and Voronin and Kolomensky (1964). They considered the possibility of accelerating charged particles in such a field, naturally, within the framework of the classical theory. The resonant amplification of an electromagnetic wave by a particle is also possible (Baier and Milstein 1977). Such an effect occurs in free electron lasers, and we have recently been informed about the construction of such lasers (Deacon *et al* 1977). Therefore, the analysis presented below may be applied to a consistent theory of such lasers. A similar mechanism may be used, apparently, for damping transverse oscillations in the beams of charged particles.

To consider the processes in external fields within the framework of quantum electrodynamics it is convenient to use the operator diagram technique developed for the case of a homogeneous electromagnetic field in the work by Baier *et al* (1975) and also for the case of an electromagnetic plane wave in the work by Baier *et al* (1976). The method is based on the operator representation of the Green function of a charged particle in the field with a subsequent disentanglement of operator expressions. Consideration of radiative effects in a given field is a considerably more complicated problem than that of the above cases and has not been attempted previously. In this paper it is solved with the help of the corresponding operator technique which turns out to be a very appropriate method in this case. Below the mass operator a scalar particle is found in the field of the configuration under consideration. This operator describes the main characteristics of behaviour of a charged particle and, at the same time, makes it possible to avoid some complications arising in the solution of the spinor particle problem which, it is assumed, is solved elsewhere.

We shall describe the electromagnetic field under study by a potential

$$\mathcal{A}_\mu = \mathcal{A}_\mu(x_\parallel) + \mathcal{A}_\mu(\phi) \tag{1}$$

where $\phi = \kappa x$, $\kappa x_\parallel = 0$. Let us assume that the magnetic field is directed towards the third axis, along which the waves propagate; then

$$\mathcal{A}^1(x_\parallel) = -x^2 H, \qquad \mathcal{A}^\mu(\phi) = n_1^\mu a_1(\phi) + n_2^\mu a_2(\phi). \tag{2}$$

Here $\phi = \kappa x = x^0 - x^3$ and the vectors $\kappa^\mu = g_0^\mu + g_3^\mu$, $n_1^\mu = g_1^\mu$, $n_2^\mu = g_2^\mu$ are introduced, where g_ν^μ are components of the metric tensor.

Let us represent the electromagnetic field strength (1) in the form

$$\mathcal{F}^{\mu\nu} = F^{\mu\nu} + \sum_{k=1}^{2} f_k^{\mu\nu} a_k'(\phi), \qquad f_k^{\mu\nu} = \kappa^\mu n_k^\nu - \kappa^\nu n_k^\mu \tag{3}$$

where $F^{21} = H$, and H is a magnetic field.

2. Mass operator

In the present approach the mass operator for the spin-0 particle can be represented as follows (Baier *et al* 1975, equation (1.10)):

$$M^{(0)} = -\frac{ie^2}{(2\pi)^4} \int d^4k (2\mathcal{P} - k)^\mu \frac{1}{(\mathcal{P} - k)^2 - m^2 + i\epsilon} (2\mathcal{P} - k)_\mu \frac{1}{k^2 + i\epsilon} \tag{4}$$

where $\mathcal{P}_\mu = i\partial_\mu - e\mathcal{A}_\mu (e > 0)$. Before the integration over k in (4) it is necessary to perform the transformation of the integrand taking account of the non-commutativity of an operator component $\mathcal{P}_\mu$. Let us carry out a standard exponential parametrisation of propagators:

$$\frac{1}{k^2 + i\epsilon} \frac{1}{(\mathcal{P} - k)^2 - m^2 + i\epsilon} = -\int_0^\infty s\, ds \int_0^1 du\, e^{-isum^2} \exp\{i[su(\mathcal{P}^2 - 2\mathcal{P}k) + sk^2]\}. \tag{5}$$

Using this representation it is possible to transform the mean value of operator (4) on the mass shell into the form:

$$\langle M^{(0)} \rangle = \left\langle \frac{ie^2}{(2\pi)^4} \int d^4k \int_0^\infty ds \left(s \int_0^1 du\, 4\mathcal{P}^\mu \exp[isu(\mathcal{P} - k)^2] \mathcal{P}_\mu \right.\right.$$
$$\left.\left. \times \exp[is(1-u)k^2 - isum^2] - i \exp\{is[(\mathcal{P} - k)^2 - m^2]\} \right) \right\rangle. \tag{6}$$

After substitution of parametrisation (5) into integral (4) both the exponential index and the coefficient of the exponential in the integrand contain variable k. We have integrated by parts the terms containing $\mathcal{P}k$ in the coefficient of the exponential in equation (6), thus enabling us to express $\langle M^{(0)}\rangle$ in a form where only the index of the exponential depends on k. The mass operator also contains terms which do not depend on external field. These terms are not found in (6) because they are dropped in the regularisation. Thus, evaluation of an integral over k is reduced to the calculation of a value:

$$Q^{(0)} = \int d^4k \exp[isu(\mathcal{P}-k)^2]\exp[is(1-u)k^2]$$
$$= \int d^4k\, e^{-ikX}\exp(isu\mathcal{P}^2)\, e^{ikX}\exp[is(1-u)k^2] \tag{7}$$

where the shift operator in momentum space is used: for some function $f(\mathcal{P})$ the following takes place: $e^{-ikX}f(\mathcal{P})\,e^{ikX}f(\mathcal{P}-k)$, $[\mathcal{P}_\mu, X_\nu] = ig_{\mu\nu}$. To calculate the integral (7) it is necessary to transform the operator expression $\exp(isu\mathcal{P}^2)$. This transformation (disentanglement), being one of the basic points in our consideration, is made in the appendix. In the calculation of integral (7) let us use the representation (A.12) for $\exp(isu\mathcal{P}^2)$. Integration over variables k^0, k^3 may be carried out using the same method as Baier *et al* (1976). In this case, it is necessary to have the variables

$$k_\phi = \tfrac{1}{2}(k^0+k^3), \qquad k_\xi = \tfrac{1}{2}(k^0-k^3).$$

Integration over k_ϕ giving $\delta(k_\xi - up_\xi)$, means that integration over k_ξ is reduced to the substitution $k_\xi \to up_\xi$. Integration over the variables k^1, k^2 can be made following the method of Baier *et al* (1975). finally we get

$$Q^{(0)} = -\frac{i\pi^2}{s^2}\frac{1}{\sqrt{D}}\, e^{i\Phi}\exp\left(i(\mathcal{P}-q)_\parallel^2 \frac{\rho}{eH}\right)\exp(i\eta\mathcal{P}_\perp^2) \tag{8}$$

where we use the following notations† (see equations (A.4) and (A.6)):

$$q = x\int_0^1 \Delta(\eta y)\, e^{-2xBy}\, dy(\cotan x + B), \qquad \eta = su(1-u),$$

$$\rho = x - a(x), \qquad a(x) = \tan^{-1}\left(\frac{1-\cos 2x}{\sin 2x + [2x(1-u)/u]}\right),$$

$$\Phi = su\left[2su\int_0^1 dy_2\int_0^{y_2} dy_1\, \Delta(\eta y_1)\exp[-2Bx(y_1-y_2)]eF\,\Delta(\eta y_2) + \int_0^1 \Delta^2(\eta y)\, dy - x\cotan x\left(\int_0^1 \Delta(\eta y)\, e^{-2Bxy}\, dy\right)^2\right], \tag{9}$$

$$D = \frac{u^2}{x^2}\left[\sin^2 x + x^2\left(\frac{1-u}{u}\right)^2 + \frac{1-u}{u}x\sin 2x\right]$$

where $x = useH$, and matrix $B^{\mu\nu} = F^{\mu\nu}/H$. The mass operator (6) includes a combination $\mathcal{P}^\mu Q^{(0)}\mathcal{P}_\mu$. It is easy to verify that (cf equation (A.1)):

$$\mathcal{P}^\mu Q^{(0)}\mathcal{P}_\mu = \tfrac{1}{2}[(\mathcal{P}^2 - \mathcal{P}_\parallel^2), Q^{(0)}] + \mathcal{P}_\parallel^\mu Q^{(0)}\mathcal{P}_\mu. \tag{10}$$

† The branches of multiple-valued functions in equation (9) should be chosen in the same way as in Baier *et al* (1975). Here and below we extensively use the matrix form for the notation, e.g., $\Delta Bq = \Delta_\mu B^{\mu\nu}q_\nu$.

Taking this into account the problem is reduced to the calculation of a combination

$$\exp[\mathrm{i}(\mathcal{P}-q)_{\parallel}^{2}\rho/eH]\exp(\mathrm{i}\eta\mathcal{P}_{\perp}^{2})\mathcal{P}_{\parallel}\exp(-\mathrm{i}\eta\mathcal{P}_{\perp}^{2})\exp[-\mathrm{i}(\mathcal{P}-q)_{\parallel}^{2}\rho/eH]$$
$$=q-\Delta(\eta)+\mathrm{e}^{-2B\rho}(\mathcal{P}_{\parallel}-q). \tag{11}$$

Using (11) we have, on the mass shell,

$$\mathcal{P}^{\mu}Q^{(0)}\mathcal{P}_{\mu}=\{m^{2}+\mathrm{i}eH\sin 2\rho-\tfrac{1}{2}[(1-\mathrm{e}^{-2B\rho})(\mathcal{P}_{\parallel}-q)+\Delta(\eta)]^{2}\}Q^{(0)}. \tag{12}$$

To calculate $\langle M^{(0)}\rangle$ (equation (6)) it is also necessary to find the second term in the large parentheses. It is not difficult to make sure that the integral $\int \mathrm{d}^4k$ from this expression is given by $-\mathrm{i}Q^{(0)}(u=1)\,\mathrm{e}^{-\mathrm{i}sm^2}$. Substituting this result in (12) and (6) we obtain the mean (from the mass operator) value on the mass shell. The renormalisation of this value is reduced to the subtraction of its value at the field $\mathcal{F}_{\mu\nu}=0$ (see Baier *et al* 1975, 1976). As a result we get the mean value of the renormalised mass operator for the spin-0 particle $\langle M^{(0)}\rangle = M_1+M_2$ with

$$M_2=\frac{\mathrm{i}\alpha}{4\pi}\int_0^{\infty}\frac{\mathrm{d}s}{s^2}\exp(-\mathrm{i}sm^2)\Big(1-\frac{eHs}{\sin(eHs)}\Big)$$
$$M_1=\frac{\alpha}{\pi}\Big\langle\int_0^{\infty}\frac{\mathrm{d}s}{s}\int_0^1\mathrm{d}u\Big[\{m^2+\mathrm{i}eH\sin 2\rho-\tfrac{1}{2}[(1-\mathrm{e}^{-2B\rho})(\mathcal{P}_{\parallel}-q)+\Delta(\eta)]^2\}\frac{\mathrm{e}^{\mathrm{i}\Phi}}{\sqrt{D}} \tag{13}$$
$$\times\exp\Big(\mathrm{i}(\mathcal{P}-q)_{\parallel}^{2}\frac{\rho}{eH}\Big)\exp(\mathrm{i}\eta\mathcal{P}_{\perp}^{2}-\mathrm{i}sum^2)-m^2\exp(-\mathrm{i}su^2m^2)\Big]\Big\rangle$$

where the integration contour passes below the real axis. The obtained expression (13) in the extreme case $H=0(\rho=0)$ is directly converted into the mean value of the mass operator in an electromagnetic plane wave field. (See Baier *et al* (1976), equation (2.22), where it should be taken into account that for the function β (2.13) one has $\langle \mathrm{d}\beta/\mathrm{d}\phi\rangle=0$.) The result following from (13) for a magnetic field in the limit $a_{1,2}=0$ is a new representation of the mass operator in this field. After the transformation of the latter using integration by parts it is possible to show that this representation is the same as that previously obtained (see Baier *et al* 1975, equation (2.41)).

3. Mean value of mass operator

The solution of the Klein–Gordon equation for the field configuration under consideration was found by Redmond (1965). We represent it in the following form:

$$\Psi_n=\exp\Big[-\mathrm{i}\Big(\frac{\lambda}{2}\xi+\frac{m^2+b_n}{2\lambda}\phi-\tfrac{1}{2}\int_{-\infty}^{\phi}e\mathcal{A}(\phi')\dot{K}(\phi')\,\mathrm{d}\phi'+K(\phi)\pi\Big)\Big]\psi_n(x_{\parallel}) \tag{14}$$

where

$$\pi_\mu=\mathrm{i}\partial_\mu-e\mathcal{A}_\mu(x_{\parallel}),\qquad \xi=x^0+x^3,\qquad p_\xi\Psi_n=\frac{\lambda}{2}\Psi_n,\qquad b_n=eH(2n+1),$$
$$K(\phi)=\mathrm{e}^{eF\phi/\lambda}\int_{-\infty}^{\phi}\mathrm{e}^{-eF\phi'/\lambda}e\mathcal{A}(\phi')\frac{\mathrm{d}\phi'}{\lambda},\qquad \dot{K}(\phi)\equiv\mathrm{d}K(\phi)/\mathrm{d}\phi, \tag{15}$$
$$\psi_n(x_{\parallel})=\Big(\frac{eH}{\pi}\Big)^{1/4}(2^n n!)^{-1/2}\exp(\mathrm{i}p_x x)\exp\Big[-\frac{eH}{2}\Big(y+\frac{p_x}{eH}\Big)^2\Big]H_n\Big(\sqrt{eH}\Big(y+\frac{p_x}{eH}\Big)\Big)$$

where H_n are the Hermite polynomials; for the sake of convenience we use x, y, z for the notations of vector components instead of x^1, x^2, x^3, respectively. Using the above solution of the Klein–Gordon equation, equations (A.10), (A.12) and also the relations

$$\mathcal{P}^2\Psi_n = m^2\Psi_n, \qquad \pi_\parallel^2\psi_n = -b_n\psi_n,$$
$$e^{iK\pi}f(\mathcal{P}_\parallel)\,e^{-iK\pi} = f(\mathcal{P}_\parallel - eFK), \qquad \lambda\dot{K} = e\mathcal{A}(\phi) + eFK,$$
$$\exp[i\eta(\pi_\parallel-\beta)^2] = \exp(i\eta\pi_\parallel^2)\exp\left(-2i\eta\beta\int_0^1 e^{2eF\eta y}\,dy\,\pi_\parallel\right)e^{i\nu}, \tag{16}$$
$$\nu = \beta^2\eta + 2\beta\eta^2\int_0^1 dy_2\int_0^{y_2} dy_1\exp[2eF\eta(y_2-y_1)]eF\beta = \beta^2\frac{\sin(2eH\eta)}{2eH},$$

where $f(\mathcal{P}_\parallel)$ is some function, η is a parameter and $\beta = \beta(x_\parallel)$ is an arbitrary 4-vector, we come to the following expression for M_1 (equation (13)):

$$M_1 = \frac{\alpha}{\pi}\int_0^\infty\frac{ds}{s}\int_0^1 du\,\exp(-isu^2m^2)\int d\phi\,dx\,dy\Big[\psi_n^*(x_\parallel)\Big((\sigma_0+\sigma\pi_\parallel)$$
$$\times\frac{1}{\sqrt{D}}e^{i\Xi}\,e^{-i\chi\pi}\,e^{i\tau\pi} - m^2\Big)\psi_n(x_\parallel)\Big]. \tag{17}$$

Here the following notation is introduced:

$$\chi = \frac{2\rho}{eH}\int_0^1 e^{-2B\rho y}\,dy(q+\lambda\dot{K}) = \frac{1-e^{-2B\rho}}{eF}(q+\lambda\dot{K}),$$
$$\tau = \frac{1-e^{-2eF\eta}}{eF}(q_\eta+\lambda\dot{K}),$$
$$\sigma = (1-e^{-2B\rho})(\Delta(\eta)-eF\chi), \tag{18}$$
$$\sigma_0 = m^2 + ieH\sin 2\rho + 2b_n\sin^2\rho - \tfrac{1}{2}(\Delta(\eta)-eF\chi)^2,$$
$$\Xi = \Phi - \Phi(\eta) + \Phi_1 + \Phi_2,$$
$$\Phi_1 = \frac{eH}{4}(\chi^2\operatorname{cotan}\rho - \tau^2\operatorname{cotan} eH\eta), \qquad \Phi_2 = b_n\left(\frac{eH\eta-\rho}{eH}\right)$$

where χ, τ, σ are 4-vectors, q_η is given by equation (A.11), $\Phi(\eta)$ by equation (A.13), and q, Φ are given by equation (9), i.e. q and q_η, Φ and $\Phi(\eta)$ should be distinguished.

For further calculations let us take into account that the function ψ_n satisfies the relation

$$\sigma\pi\psi_n = -\sqrt{eH}\left[\left(\frac{n+1}{2}\right)^{1/2}\sigma^+\psi_{n+1} + \left(\frac{n}{2}\right)^{1/2}\sigma^-\psi_{n-1}\right], \tag{19}$$

where $\sigma^\pm = \sigma^x \pm i\sigma^y$. Let us transform the operators $e^{-i\chi\pi}$ and $e^{i\tau\pi}$ appearing in (17) as follows:

$$e^{i\tau\pi} = \exp(i\tau^x\pi_x)\exp(i\tau^y\pi_y)\exp(\tfrac{1}{2}\tau^x\tau^y[\pi_x,\pi_y]) = \exp(i\tau^x\pi_x + \tfrac{1}{2}ieH\tau^x\tau^y)\exp(i\tau^y\pi_y). \tag{20}$$

Note that $\exp(i\tau^y\pi_y)$ is the shift operator in the y axis. Taking account of (19), (20) and also of the integral (Gradstein and Ryzhik 1962):

$$\int_{-\infty}^{+\infty} e^{-x^2} H_m(x+y) H_n(x+z)\,dx = 2^n\sqrt{\pi}\, m!\, z^{n-m} L_m^{n-m}(-2yz) \qquad (m \leqslant n), \tag{21}$$

where L_m^{n-m} are the Laguerre polynomials, let us carry out the integration over x, y:

$$M_1 = \frac{\alpha}{\pi}\int_0^\infty \frac{ds}{s}\int_0^1 du\, \exp(-isu^2m^2)\Big[\int d\phi\Big(\sigma_0 L_n(\theta) - \frac{ieH}{2}[\sigma^+(\chi^- - \tau^-)L_{n-1}^1(\theta)$$

$$+\sigma^-(\chi^+ - \tau^+)L_n^1(\theta)]\Big)\frac{1}{\sqrt{D}}\exp\Big(i\Xi_1 - \frac{\theta}{2}\Big) - m^2\Big]. \tag{22}$$

Here

$$\Xi_1 = \Xi + \frac{eH}{2}(\tau B\chi), \qquad \theta = -\frac{eH}{2}(\chi-\tau)^2 \geqslant 0,$$
$$\chi^\pm = \chi^x \pm i\chi^y, \qquad \tau^\pm = \tau^x \pm i\tau^y. \tag{23}$$

For the remaining notation see equations (9) and (18). While averaging in M_2 (see (13)) normalisation integrals appear, i.e.:

$$M_2 = \frac{i\alpha}{4\pi}\int_0^\infty \frac{ds}{s^2}\exp(-ism^2)\Big(1 - \frac{eHs}{\sin(eHs)}\Big). \tag{24}$$

Substituting the obtained results (22), (24) in $\langle M^{(0)}\rangle = M_1 + M_2$ we get the mean value of a mass operator of the scalar particle on the mass shell in an electromagnetic plane wave field of a general type propagated along the magnetic field H. To calculate an integral over ϕ it is necessary to give an explicit form of the field. The case of a monochromatic wave will now be considered.

4. Shift of quasi-energy

To reveal the sense of the mean value of the mass operator in a time-dependent field let us consider the modified Klein–Gordon equation, proposed by Schwinger, with radiative corrections taken into account:

$$(\mathscr{P}_0^2 - \boldsymbol{\mathscr{P}}^2 - m^2 - M^{(0)})\Psi = 0. \tag{25}$$

In the case when the potential is a periodic time function its solution can be represented as follows:

$$\Psi = e^{-i\epsilon t} f(\boldsymbol{x}, t) \tag{26}$$

where ϵ is the quasi-energy of a particle and f is the periodic function (with the same period). Let us represent the solution of equation (25) in zero order in the following form:

$$\Psi_0 = e^{-i\epsilon_0 t} f_0(\boldsymbol{x}, t). \tag{27}$$

Multiplying equation (25) on the left by Ψ_0^+ and integrating over the space coordinates we get

$$\int (\Psi_0^+\,\partial_0^2\Psi - \Psi\,\partial_0^2\Psi_0^+)\,d^3x = -\int \Psi_0^+ M^{(0)}\Psi\,d^3x. \tag{28}$$

Let us evaluate the integral from (28) over a single period of time, bearing in mind that in the lowest order of perturbation theories the difference $\epsilon-\epsilon_0$ and $M^{(0)}$ are the values of the order α. Then discarding the higher-order term we find†

$$\Delta\epsilon \equiv \epsilon-\epsilon_0=\frac{1}{2\langle\mathscr{P}_0\rangle}\int d^4x\,\Psi_0^+ M^{(0)}\Psi_0 \tag{29}$$

where $\langle\mathscr{P}_0\rangle=\frac{1}{2}\mathrm{i}\int d^4x(\Psi_0^+\overleftrightarrow{\partial}_0\Psi_0)$ is a zero component of the mean kinetic momentum of a particle (at a given normalisation of wavefunctions). This result determines a physical sense of the mean value of the mass operator. Let us represent

$$\Delta\epsilon=\mathrm{Re}\,\Delta\epsilon-\tfrac{1}{2}\mathrm{i}W, \tag{30}$$

where W is the mean value over the period probability of radiation, by a particle in a given field per unit time. Substituting (30) into (29) we have

$$W=-\mathrm{Im}\langle M^{(0)}\rangle/\langle\mathscr{P}_0\rangle \tag{31}$$

where $\langle M^{(0)}\rangle\equiv\int d^4x\,\Psi_0^+M^{(0)}\Psi_0$.

5. The case of a monochromatic wave

Let us now consider the case of a monochromatic wave. The elliptically polarised monochromatic plane wave can be described by a potential

$$e\mathscr{A}^\mu(\phi)=n_1^\mu\frac{\zeta_1+\zeta_2}{2}\cos\omega\phi+n_2^\mu\frac{\zeta_1-\zeta_2}{2}\sin\omega\phi. \tag{32}$$

Helicity unit vectors prove to be very convenient for this case:

$$\epsilon=\frac{n_1+\mathrm{i}n_2}{\sqrt{2}},\qquad \epsilon^*=\frac{n_1-\mathrm{i}n_2}{\sqrt{2}}. \tag{33}$$

Then $B\epsilon=\mathrm{i}\epsilon$, $B\epsilon^*=-\mathrm{i}\epsilon^*$. In these terms a vector $K(\phi)$ (see (15)) for a field (32) has the following form:

$$K(\phi)=\frac{\mathrm{i}}{(2\sqrt{2})eH}\left[\zeta_1\left(\frac{\epsilon\,\mathrm{e}^{-\mathrm{i}\omega\phi}}{\nu+1+\mathrm{i}0}-\frac{\epsilon^*\mathrm{e}^{\mathrm{i}\omega\phi}}{\nu+1-\mathrm{i}0}\right)-\zeta_2\left(\frac{\epsilon\,\mathrm{e}^{\mathrm{i}\omega\phi}}{\nu-1-\mathrm{i}0}-\frac{\epsilon^*\,\mathrm{e}^{-\mathrm{i}\omega\phi}}{\nu-1-\mathrm{i}0}\right)\right] \tag{34}$$

where $\nu\equiv\omega\lambda/eH$. Similarly a vector q (see (9)) is

$$q=\frac{1}{2\sqrt{2}}[\epsilon(c_+\zeta_1\,\mathrm{e}^{-\mathrm{i}\omega\phi}+c_-\zeta_2\,\mathrm{e}^{\mathrm{i}\omega\phi})+\mathrm{HC}] \tag{35}$$

where

$$c_\pm=\frac{x\,\sin(x\pm y)}{(x\pm y)\sin x}\mathrm{e}^{\pm\mathrm{i}y}-1,\qquad x=sueH,\qquad y=\omega\lambda\eta. \tag{36}$$

According to the values found we may construct the vectors χ and τ in terms of which the answer is given (see (22)). Let us still represent an explicit form of the function Φ using (13) and (9):

$$\Phi=\frac{x}{4eH}[\zeta_1^2V_1+\zeta_2^2V_2+\zeta_1\zeta_2\cos 2(\omega\phi-y)V_3] \tag{37}$$

† The shift of quasi-energy at the fixed quasi-momentum is considered.

where

$$V_1 = \frac{y}{x+y}\left(\frac{\sin(x+y)}{x+y}\frac{\sin y}{y}\frac{x}{\sin x} - 1\right), \qquad V_2 = V_1(y \to -y),$$
$$V_3 = \frac{\sin 2y}{x^2 - y^2}(y - x \operatorname{cotan} x \tan y). \tag{38}$$

The argument of the Laguerre polynomials is given by:

$$\theta = \frac{1}{2eH}|\Pi_1 \mathrm{e}^{\mathrm{i}(\omega\phi - y)} + \Pi_2 \mathrm{e}^{-\mathrm{i}(\omega\phi - y)}|^2 \tag{39}$$

where

$$\Pi_1 = \frac{\zeta_1}{\sin x}\left(\frac{x}{x+y}\sin(x+y)\sin\rho - \frac{1}{\nu+1}\sin(\rho+y)\sin x\right)$$
$$\Pi_2 = \Pi_1(\zeta_1 \to \zeta_2, y \to -y, \nu \to -\nu). \tag{40}$$

For the elliptically polarised wave the terms $\zeta_1\zeta_2$ in the exponent index in (22) and in the argument of the Laguerre polynomials (see (39)) depend explicitly on ϕ, this dependence entering only as a linear function of $\cos 2(\omega\phi - y)$. Therefore, for the integration over ϕ in (22) it is necessary to evaluate integrals of the type

$$\int_0^{2\pi} \mathrm{e}^{\alpha\cos 2\phi} L_n(a + b\cos 2\phi)\,\mathrm{d}\phi = \frac{2\pi}{n!}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[(s-1)^n \mathrm{e}^{sa} I_0(\alpha + sb)]_{s=0} \tag{41}$$

where I_0 is the modified Bessel function of the first kind. Using (41) one can obtain an explicit form of the integral over ϕ in $\langle M^{(0)}\rangle = M_1 + M_2$ (see (22), (24)) for a monochromatic wave with elliptical polarisation.

In the following we restrict ourselves to the analysis of the case of a circularly polarised wave $\zeta_1 = \zeta$, $\zeta_2 = 0$, the standard parameter of the wave intensity being

$$\xi^2 = \zeta^2/4m^2. \tag{42}$$

In this case the integrand in (22) is independent of ϕ; therefore, the integral over ϕ is reduced to the normalisation integral. Thus, the mean value of a mass operator of the scalar particle in the field of a circularly polarised wave propagated along the magnetic field H is:

$$\langle M^{(0)}\rangle = \frac{\alpha}{\pi}m^2 \int_0^\infty \frac{\mathrm{d}s}{s}\left\{\int_0^1 \mathrm{d}u \exp(-\mathrm{i}su^2m^2)\left[\frac{Z}{\sqrt{D}}\exp\left(\mathrm{i}\Phi_c - \frac{\theta}{2}\right) - 1\right]\right.$$
$$\left. + \frac{\mathrm{i}}{4sm^2}\exp(-\mathrm{i}sm^2)\left(1 - \frac{eHs}{\sin(eHs)}\right)\right\} \tag{43}$$

where

$$Z = Z_0 L_n(\theta) + \xi^2 Z_1, \qquad \Phi_c = \Phi_0 + \xi^2\Phi_1,$$
$$Z_0 = 1 + 2\mathrm{i}(H/H_0)\sin 2\rho + 2(H/H_0)(2n+1)\sin^2\rho,$$
$$Z_1 = 2|R - \sin y|^2 L_n(\theta) + 4i\sin\rho(NL^1_{n-1}(\theta) + N^* L^1_n(\theta)),$$
$$\Phi_0 = (2n+1)[x(1-u) - \rho], \tag{44}$$

$$\Phi_1=\frac{H_0}{H}\left[\left(\frac{x}{x+y}\right)^2\frac{\sin y}{\sin x}\sin(x+y)-\frac{\sin 2y}{2(1+\nu)^2}-\frac{\nu xyu}{(x+y)(1+\nu)}\right.$$
$$\left.-\operatorname{cotan}\rho|R|^2-\frac{2\,\mathrm{Im}\,R}{1+\nu}\sin y\right],$$
$$R=\mathrm{e}^{\mathrm{i}\rho}\sin\rho\left(\frac{x}{x+y}\,\frac{\sin(x+y)}{\sin x}-\frac{\mathrm{e}^{\mathrm{i}y}}{1+\nu}\right),$$
$$N=\mathrm{e}^{\mathrm{i}\rho}(R-\sin y)\left(R^*-\frac{\sin y}{1+\nu}\right),\qquad \theta=2\xi^2\frac{H_0}{H}\frac{(\mathrm{Im}\,R)^2}{\sin^2\rho}.$$

Here $x=sueH$, $y=\omega\lambda\eta$, $\nu=\omega\lambda/eH$ and $H_0=m^2/e$; for the remaining notation see (9).

6. Extreme cases, cyclotron resonance

The obtained result (43) gives a general picture of radiative effects (probability of radiation, level shifts) in a field of the configuration under consideration when a monochromatic wave has a circular polarisation. Let us now consider $\langle M^{(0)}\rangle$ (given by (43)) in a number of extreme cases.

When $H/H_0\ll 1$, $\nu=\lambda\omega/eH\gg 1$ we have a particle in the intense electromagnetic wave and a relatively weak magnetic field. In this case the main contribution in the integral (43) comes from the region of small x. Carrying out the corresponding expansions and retaining the terms linear with respect to $1/\nu$ (the expansion in H/H_0 begins with quadratic terms) we get

$$\langle M^{(0)}\rangle=\frac{\alpha}{\pi}m^2\int_0^\infty\frac{\mathrm{d}y}{y}\int_0^\infty\frac{\mathrm{d}v}{(1+v)^2}\exp\left(-\frac{\mathrm{i}vy}{\Lambda}\right)\left(\exp\left[\frac{\mathrm{i}\xi^2vy}{\Lambda}\left(\frac{\sin^2y}{y^2}-1\right)\right]\left\{(1+2\xi^2\sin^2y)\right.\right.$$
$$\left.\left.\times\left[1+\frac{\mathrm{i}\xi^2yv}{\Lambda\nu}\left(2+v+v\frac{\sin 2y}{2y}-2(1+v)\frac{\sin^2y}{y^2}\right)\right]-\frac{4\xi^2}{\nu}\sin^2y\right\}-1\right)\tag{45}$$

where $\Lambda=\omega\lambda/m^2$, and the variables $y=\lambda\omega\eta$, $v=u/(1-u)$ are used. At $\nu\to\infty$ (45) is converted into an expression for the mean value of the mass operator in the intense, circularly polarised wave field (see, e.g., Baier *et al* 1976, equation (2.30)); terms $1/\nu$ represent the corrections owing to the presence of the magnetic field.

In the case $\xi\ll 1$ we have the description of processes in a magnetic field of arbitrary strength in the presence of a weak plane wave. Then, to obtain an explicit expression for $\langle M^{(0)}\rangle$, it is necessary to make the following substitutions in equation (43):

$$Z\to Z_0+\xi^2Z_2,\qquad \mathrm{i}\Phi_c-\tfrac{1}{2}\theta\to\mathrm{i}\Phi_0\tag{46}$$

where

$$Z_2=2|R-\sin y|^2+8\mathrm{i}\sin\rho\;\mathrm{Re}\,N+\mathrm{i}Z_0\Phi_1-(2n+1)\frac{H_0}{H}Z_0\left(\frac{\mathrm{Im}\,R}{\sin\rho}\right)^2;\tag{47}$$

for the remaining notation see (44). A term Z_0 in Z (see (46)) describes radiative effects in the magnetic field. After the division by a flux $\lambda/\langle\mathscr{P}_0\rangle$ the term $\xi^2 Z_2$ substituted into equation (31) gives a total cross section for the Compton scattering in the magnetic field of arbitrary strength:

$$\sigma = -\frac{4\alpha^2}{m^2\Lambda}\,\mathrm{Im}\int_0^\infty \frac{ds}{s}\int_0^1 du\,\exp(-isu^2m^2)\frac{e^{i\Phi_0}}{\sqrt{D}}Z_2. \tag{48}$$

The Compton effect has been discussed previously by De Raad *et al* (1974) who, within the framework of the Schwinger operator technique for a scalar particle in the magnetic field, succeeded in taking account of the interaction with the electromagnetic monochromatic wave propagated along a field H in a lower-order perturbation theory. Expression (48) obtained here is essentially more compact than that used by De Raad *et al* (1974).

In the case $H/H_0 \ll 1$ and $\nu \gg 1$ it is possible to obtain an explicit expression for a cross section σ. Then the contribution in the integral (48) comes from the region of small x. Carrying out the expansions we find, after the evaluation of the integrals,

$$\sigma = \sigma_0 + \sigma_H. \tag{49}$$

Here

$$\sigma_0 = \frac{2\pi\alpha^2}{m^2\Lambda}\left[\frac{2(1+\Lambda)^2}{\Lambda(1+2\Lambda)} - \frac{1}{\Lambda}\left(1+\frac{1}{\Lambda}\right)\ln(1+2\Lambda)\right],$$
$$\sigma_H = \frac{2\pi\alpha^2}{m^2\Lambda\nu}\left[\frac{2}{\Lambda} - \frac{2\Lambda}{1+2\Lambda} - \frac{1}{\Lambda}\left(1+\frac{1}{\Lambda}\right)\ln(1+2\Lambda)\right] \tag{50}$$

where

$$\Lambda = \frac{\omega(\langle\mathscr{P}_0\rangle - \langle\mathscr{P}^3\rangle)}{m^2}, \qquad \langle\mathscr{P}_0\rangle^2 = \langle\mathscr{P}^3\rangle^2 + m^2 + eH(2n+1).$$

Value σ_0 coincides in its form with the cross section of Compton scattering of a scalar particle in the absence of a magnetic field if $\Lambda = kp/m^2$; k, p are momenta of the photon and particle; σ_H is the correction owing to the presence of a magnetic field. Averaging over polarisations of the photon (in the case of a linear polarisation as well) this correction vanishes. For this reason the latter does not appear in the work by De Raad *et al* (1974) where the case of unpolarised photons was considered.

As has been noted already in the field configuration under consideration a resonant situation occurs when the field frequency coincides with the cyclotron frequency of a particle in the magnetic field; $\nu = -1$. We will consider this question in the case when $H/H_0 \ll 1$, $\nu \sim 1$ $(\Lambda = \lambda\omega/m^2 \ll 1)$. Then the expression $\langle M^{(0)}\rangle$ (see (43)) can be transformed into the following form:

$$\langle M^{(0)}\rangle = \frac{\alpha}{\pi}m^2\int_0^\infty \frac{dy}{y}\int_0^\infty \frac{dv}{(1+v)^2}\exp\left(-\frac{ivy}{\Lambda}\right)\left\{\left(1+\frac{2\xi^2\nu^2}{\delta^2}\sin^2 y\right)\right.$$
$$\left.\times\exp\left[\frac{ivy}{\Lambda}\frac{\xi^2\nu^2}{\delta^2}\left(\frac{\sin^2 y}{y^2}-1\right)\right]-1\right\} \tag{51}$$

where $\delta = |1+\nu|$.

For completeness let us give a value of the mean $\langle\mathcal{P}^0\rangle$, $\langle\mathcal{P}^3\rangle$ if a circularly polarised monochromatic wave propagates along the magnetic field

$$\langle\mathcal{P}^0\rangle=\frac{\lambda}{2}+\frac{m^2+eH(2n+1)}{2\lambda}+\frac{\nu^4m^6}{2\lambda^5}\frac{\xi^2H^2}{\delta^2H_0^2},$$
$$\langle\mathcal{P}^3\rangle=\langle\mathcal{P}^0\rangle-\lambda. \tag{52}$$

The value $\langle\mathcal{P}^0\rangle$ enters in the expression for the probability of radiation.

The properties of the mean value of a mass operator in the vicinity of a resonance are of significant interest. It appears that these properties depend greatly on the degree of nearness to the resonance. Let us consider the value $\langle M^{(0)}\rangle$ (from (51)) in the region where $H\xi/H_0 \ll \delta \leqslant 1$. Then the main contribution in the integral over v in (51) comes from the interval $v \ll 1$. Carrying out the corresponding expansions and evaluating the integral over v we find

$$\langle M^{(0)}\rangle=-\frac{\mathrm{i}\alpha}{\pi}m^2\frac{\Lambda\xi^2\nu^2}{\delta^2}\int_0^\infty\frac{\mathrm{d}y}{y^4}\frac{\sin^2y-y^2\cos 2y}{1+(\xi^2\nu^2/\delta^2)[1-(\sin^2y/y^2)]} \tag{53}$$

so that in this region the value $\langle M^{(0)}\rangle$ is purely imaginary. In the limit $\xi\to 0$, with the procedure used to obtain equation (48), we have from (53) a cross section of the Thompson scattering of a circularly polarised wave in the magnetic field:

$$\sigma_{\mathrm{T}}=\frac{8\pi\alpha^2}{3m^2}\frac{\nu^2}{(1+\nu)^2}. \tag{54}$$

In the region $\delta \leqslant H\xi/H_0$ the main contribution in integral (51) comes from the region of small y. Expanding in the exponential function index over y and evaluating the integral we get

$$\langle M^{(0)}\rangle=\frac{\alpha}{3\pi}m^2\int_0^\infty\frac{\mathrm{d}v}{(1+v)^2}(5+2v)\left[L_{2/3}\left(\frac{2v}{3\kappa}\right)-\frac{\mathrm{i}}{\sqrt3}K_{2/3}\left(\frac{2v}{3\kappa}\right)\right] \tag{55}$$

where $\kappa=\xi H/\delta H_0$, $K_{2/3}$ is the modified Bessel function of the third kind (Macdonald function), and the function $L_{2/3}$ is determined in Baier *et al* (1973, p 181).

Let us give here asymptotic expansions of equation (55) (cf Baier *et al* 1973 as well):

$$\langle M^{(0)}\rangle=\alpha m^2\kappa\left(\frac{8}{3\pi}\kappa\ln\frac{1}{\kappa}-\frac{\mathrm{i}5}{2\sqrt3}\right)\qquad \kappa\ll 1$$
$$\langle M^{(0)}\rangle=\frac{4}{9}\alpha m^2\Gamma\left(\frac{2}{3}\right)(3\kappa)^{2/3}\left(\frac{1}{\sqrt3}-\mathrm{i}\right)\qquad \kappa\gg 1. \tag{56}$$

Let us discuss the results obtained. In the region where $H\xi/H_0 \ll \delta \leqslant 1$ we are relatively far from the resonance point; in particular, the resonant contribution in $\langle\mathcal{P}^0\rangle$ (see (52)) is not yet dominant. When one approaches the resonance (as δ is decreasing) the value $|\langle M^{(0)}\rangle|$ becomes larger. The probability of radiation (see (31), (52)) increases too. At $\delta \ll 1$ we have from (53):

$$M^{(0)}=-\mathrm{i}\frac{5}{2\sqrt3}\alpha m^2\frac{\xi H}{\delta H_0}. \tag{57}$$

In the region where $\delta \leqslant H\xi/H_0$ we are quite near the resonance point. Then the mean value of a zero component of the kinetic momentum increases in a resonant manner as

δ is decreased (see (52)), the resonant contribution in $\langle\mathcal{P}^0\rangle$ dominating. Therefore, particle motion becomes quasi-classical. For this reason expression (55) coincides with the scalar particle mass operator calculated in a quasi-classical approximation (see Baier *et al* 1973, p 188). However, instead of the characteristic parameter $\chi = H\langle\mathcal{P}^0\rangle/H_0 m$ a different one $\kappa = \xi H/\delta H_0$ is included in (57). The mass operator increases as it approaches resonance (see (56)) but the radiation probability (31) decreases owing to the faster increase of the value of $\langle\mathcal{P}^0\rangle$ in the denominator.

Appendix

Let us consider an operator $\exp(\mathrm{i}s\mathcal{P}^2)$ represented in the following form (cf appendix to Baier *et al* 1976):

$$e^{\mathrm{i}s\mathcal{P}^2} = e^{\mathrm{i}s(a+b)} = L(s)\, e^{\mathrm{i}sb}\, e^{\mathrm{i}sa} \tag{A.1}$$

where

$$a \equiv \mathcal{P}_0^2 - \mathcal{P}_3^2 \equiv \mathcal{P}_\perp^2, \qquad b \equiv -(\mathcal{P}_1^2 + \mathcal{P}_2^2) \equiv \mathcal{P}_\parallel^2.$$

Differentiating (A.1) with respect to s and multiplying the result on the left-hand side by L^{-1} and on the right-hand side by $e^{-\mathrm{i}sa}\, e^{-\mathrm{i}sb}$ we get

$$\mathrm{i}L^{-1}\frac{\mathrm{d}L}{\mathrm{d}s} = b(\phi) - e^{\mathrm{i}sb(\phi)} f(s)\, e^{-\mathrm{i}sb(\phi)} \tag{A.2}$$

where $f(s) = e^{\mathrm{i}sa} b(\phi)\, e^{-\mathrm{i}sa}$. We use variables $\phi = x^0 - x^3$, and $\xi = x^0 + x^3$; then

$$\mathcal{P}_\perp^2 = 4p_\xi p_\phi \tag{A.3}$$

where $p_\phi = \mathrm{i}\,\partial/\partial\phi$, $p_\xi = \mathrm{i}\,\partial/\partial\xi$. It is now obvious that $e^{\mathrm{i}sa}$ is the shift operator over a variable ϕ so that

$$f(s) = b(\tilde{\phi}_s), \qquad \tilde{\phi}_s = \phi - 4p_\xi s. \tag{A.4}$$

Let us write down an operator $\mathcal{P}_\parallel$ in the form $\mathcal{P}_\parallel = \mathrm{i}\,\partial_\parallel - e\mathcal{A}(x_\parallel) - e\mathcal{A}(\phi) = \pi_\parallel - e\mathcal{A}(\phi)$. Taking account of this and (A.4) it is possible to transform equation (A.2) into the following form:

$$\mathrm{i}L^{-1}\frac{\mathrm{d}L}{\mathrm{d}s} = -\Delta^2 + 2\Delta\, e^{-2eFs}\mathcal{P}_\parallel \tag{A.5}$$

where the matrix form of notation is used, for example, $\mathcal{A}F\mathcal{P} = \mathcal{A}_\mu F^{\mu\nu}\mathcal{P}_\nu$,

$$\Delta_\mu(s) = e(\mathcal{A}_\mu(\tilde{\phi}_s) - \mathcal{A}_\mu(\phi)). \tag{A.6}$$

In deducing (A.5) the commutator $[\pi_\parallel^i, \pi_\parallel^k] = -\mathrm{i}eF^{ik}$ is taken into account. The solution of equation (A.5) may be written as follows:

$$L = \exp\left(\mathrm{i}s\int_0^1 \Delta^2(sy)\,\mathrm{d}y\right) T^{(-)} \exp\left(-2\mathrm{i}s\int_0^1 \Delta(sy)\, e^{-2eFsy}\mathcal{P}_\parallel\,\mathrm{d}y\right) \tag{A.7}$$

where the symbol $T^{(-)}$ denotes an anti-chronological product over 'time' y. The $T^{(-)}$ product included in (A.7) can be calculated explicitly since the commutator of operators in the index of the exponent is c-number (cf Baier *et al* 1973, § 6.3). It is easy to verify that:

$$T^{(-)}\exp\Big(\int_0^1 B(s)\,\mathrm{d}s\Big)$$
$$=\exp\Big(\int_0^1 B(s)\,\mathrm{d}s\Big)\exp\Big[\frac{1}{2}\int_0^1 \mathrm{d}s_1\int_0^1 \mathrm{d}s_2\theta(s_2-s_1)[B(s_1),B(s_2)]. \qquad \text{(A.8)}$$

Substituting (A.8) into (A.7) and then into (A.1) we get

$$\exp(\mathrm{i}s\mathscr{P}^2)=\exp\Big(\mathrm{i}s\int_0^1 \Delta^2(sy)\,\mathrm{d}y\Big)\exp\Big(-2\mathrm{i}s\int_0^1 \Delta(sy)\,\mathrm{e}^{-2eFsy}\mathscr{P}\,\mathrm{d}y\Big)$$
$$\times\exp\Big(2\mathrm{i}s^2\int_0^1 \mathrm{d}y_1\int_0^1 \mathrm{d}y_2\theta(y_2-y_1)\Delta(sy_1)\,\mathrm{e}^{2eFs(y_2-y_1)}eF\Delta(sy_2)\Big)$$
$$\times\exp(\mathrm{i}s\mathscr{P}_\parallel^2)\exp(\mathrm{i}s\mathscr{P}_\perp^2). \qquad \text{(A.9)}$$

In what follows it is convenient to transform (A.9) into a form not containing linear terms over $\mathscr{P}$. To this end let us use an expression

$$\exp[\mathrm{i}s(\mathscr{P}-q)_\parallel^2]$$
$$=\exp\Big[\mathrm{i}\Big(q^2s+2s^2\int_0^1 \mathrm{d}y_2\int_0^1 \mathrm{d}y_1 q\exp[2eFs(y_2-y_1)]eFq\Big)\Big]$$
$$\times\exp\Big(-2\mathrm{i}qs\int_0^1 \mathrm{e}^{-2eFsy}\mathscr{P}\,\mathrm{d}y\Big)\exp(\mathrm{i}s\mathscr{P}_\parallel^2). \qquad \text{(A.10)}$$

If we let

$$q=q_s=\frac{\int_0^1 \Delta(sy)\,\mathrm{e}^{-2eFsy}\,\mathrm{d}y}{\int_0^1 \mathrm{e}^{-2eFsy}\,\mathrm{d}y}=\int_0^1 \Delta(sy)\,\mathrm{e}^{-2eFsy}\,\mathrm{d}y\,(Fes+eHs\operatorname{cotan} eHs) \qquad \text{(A.11)}$$

we arrive at the following representation for $\exp(\mathrm{i}s\mathscr{P}^2)$:

$$\exp(\mathrm{i}s\mathscr{P}^2)=\mathrm{e}^{\mathrm{i}\Phi(s)}\exp[\mathrm{i}s(\mathscr{P}-q_s)_\parallel^2]\exp(\mathrm{i}s\mathscr{P}_\perp^2) \qquad \text{(A.12)}$$

where

$$\Phi(s)=2s^2\int_0^1 \mathrm{d}y_2\int_0^{y_2}\mathrm{d}y_1\,\Delta(sy_1)\exp[2eFs(y_2-y_1)]eF\,\Delta(sy_2)$$
$$+s\int_0^1 \Delta^2(sy)\,\mathrm{d}y-eHs^2\operatorname{cotan} eHs\Big(\int_0^1 \Delta(sy)\,\mathrm{e}^{-2eFsy}\,\mathrm{d}y\Big)^2. \qquad \text{(A.13)}$$

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